

# Seven Principles of Quantum Mechanics

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## Abstract

The list of basic axioms of quantum mechanics as it was formulated by von Neumann includes only the mathematical formalism of the Hilbert space and its statistical interpretation. We point out that such an approach is too general to be considered as the foundation of quantum mechanics. In particular in this approach any finite-dimensional Hilbert space describes a quantum system. Though such a treatment might be a convenient approximation it can not be considered as a fundamental description of a quantum system and moreover it leads to some paradoxes like Bell's theorem. I present a list from seven basic postulates of axiomatic quantum mechanics. In particular the list includes the axiom describing spatial properties of quantum system. These axioms do not admit a nontrivial realization in the finite-dimensional Hilbert space. One suggests that the axiomatic quantum mechanics is consistent with local realism.

## INTRODUCTION

Most discussions of foundations and interpretations of quantum mechanics take place around the meaning of probability, measurements, reduction of the state and entanglement. The list of basic axioms of quantum mechanics as it was formulated by von Neumann [1] includes only general mathematical formalism of the Hilbert space and its statistical interpretation, see also [2]–[6]. From this point of view any mathematical proposition on properties of operators in the Hilbert space can be considered as a quantum mechanical result. From our point of view such an approach is too general to be called foundations of quantum mechanics. We have to introduce more structures to treat a mathematical scheme as quantum mechanics.

These remarks are important for practical purposes. If we would agree about the basic axioms of quantum mechanics and if one proves a proposition in this framework then it could be considered as a quantum mechanical result. Otherwise it can be a mathematical result without immediate relevance to quantum theory. An important example of such a case is related with Bell's inequalities. It is known that the correlation function of two spins computed in the four-dimensional Hilbert space does not satisfy the Bell inequalities. This result is often interpreted as the proof that quantum mechanics is inconsistent with Bell's inequalities. However from the previous discussion it should be clear that such a claim is justified only if we agree to treat the four-dimensional Hilbert space as describing a physical quantum mechanical system. In quantum information theory qubit, i.e. the two-dimensional Hilbert space, is considered as a fundamental notion.

Let us note however that in fact the finite-dimensional Hilbert space should be considered only as a convenient approximation for a quantum mechanical system and if we want to investigate fundamental properties of quantum mechanics then we have to work in an infinite-dimensional Hilbert space because only there the condition of locality in space and time can be formulated. There are such problems where we can not reduce the infinite-dimensional Hilbert space to a finite-dimensional subspace.

We shall present a list from seven axioms of quantum mechanics. The axioms are well known from various textbooks but normally they are not combined together. Then, these axioms define an axiomatic quantum mechanical framework. If some proposition is proved in this framework then it could be considered as an assertion in axiomatic quantum mechanics. Of course, the list of the axioms can be discussed but I feel that if we fix the list it can help to clarify some problems in the foundations of quantum mechanics.

For example, as we shall see, the seven axioms do not admit a nontrivial realization in the four-dimensional Hilbert space. This axiomatic framework requires an infinite-dimensional Hilbert space. One can prove that Bell's inequalities might be consistent with the correlation function of the localized measurements of spin computed in the infinite-dimensional Hilbert space [16, 20]. Therefore in this sense we can say that axiomatic quantum mechanics is consistent with Bell's inequalities and with local realism. It is well known that there are no Bell's type experiments without loopholes, so there is no contradiction between Bell's inequalities, axiomatic quantum mechanics and experiments, see [21].

There is a gap between an abstract approach to the foundations and the very successful pragmatic approach to quantum mechanics which is essentially reduced to the solution of the

Schrödinger equation. If we will be able to fill this gap then perhaps it will be possible to get a progress in the investigations of foundations because in fact the study of solutions of the Schrödinger equation led to the deepest and greatest achievements of quantum mechanics.

In this note it is proposed that the key notion which can help to build a bridge between the abstract formalism of the Hilbert space and the practically useful formalism of quantum mechanics is the notion of the ordinary three-dimensional space. It is suggested that the spatial properties of quantum system should be included into the list of basic axioms of quantum mechanics together with the standard notions of the Hilbert space, observables and states. Similar approach is well known in quantum field theory but it is not very much used when we consider foundations of quantum mechanics.

Quantum mechanics is essentially reduced to the solution of the Schrödinger equation. However in many discussions of the foundations of quantum mechanics not only the Schrödinger equation is not considered but even the space-time coordinates are not mentioned (see for example papers in [6]). Such views to the foundations of quantum mechanics are similar to the consideration of foundations of electromagnetism but without mentioning the Maxwell equations.

Here I present a list from seven basic postulates of quantum mechanics which perhaps can serve as a basis for further discussions. The axioms are: Hilbert space, measurements, time, space, composite systems, Bose-Fermi alternative, internal symmetries. In particular the list includes the axiom describing spatial properties of quantum system which play a crucial role in the standard formalism of quantum mechanics. Formulations of the axioms are based on the material from [1]-[20].

The main point of the note is this: quantum mechanics is a physical theory and therefore its foundations are placed not in the Hilbert space but in space and time.

## 1 Hilbert space

To a physical system one assigns a Hilbert space  $\mathcal{H}$ . The observables correspond to the self-adjoint operators in  $\mathcal{H}$ . The pure states correspond to the one-dimensional subspaces of  $\mathcal{H}$ . An arbitrary state is described by the density operator, i.e. a positive operator with the unit trace. For the expectation value  $\langle A \rangle_\rho$  of the observable  $A$  in the state described by the density operator  $\rho$ , we have the Born-von Neumann formula

$$\langle A \rangle_\rho = \text{Tr}(\rho A)$$

## 2 Measurements

Measurement is an external intervention which changes the state of the system. These state changes are described by the concept of state transformer or instrument. Let  $\{\Omega, \mathcal{F}\}$  be a measured space where  $\Omega$  is a set and  $\mathcal{F}$  is a  $\sigma$ -algebra its subsets. A state transformer  $\Gamma$  is a state transformation valued measure  $\Gamma = \{\Gamma_B, B \in \mathcal{F}\}$  on the measured space. A state transformation  $\Gamma_B$  is a linear, positive, trace-norm contractive map on the set of trace class operators in  $\mathcal{H}$ .

An ideal state transformer  $\Gamma = \{\Gamma_i, i = 1, 2, \dots\}$  associated with discrete observable  $A = \sum_{i=1}^{\infty} a_i E_i$  is given by the Dirac-von Neumann formula

$$\Gamma_i(\rho) = E_i \rho E_i$$

if it is known that the measurement outcome is a real number  $a_i$ . Here  $E_i$  is the orthogonal projection operator. Similar formulae hold for the positive operator valued measure (POVM).

### 3 Time

The dynamics of the density operator  $\rho$  and of a state  $\psi$  in the Hilbert space which occurs with passage of time is given by

$$\rho(t) = U(t)\rho U(t)^*,$$

$$\psi(t) = U(t)\psi$$

Here  $t$  is a real parameter (time),  $U(t)$  is a unitary operator satisfying the abstract Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} U(t) = H(t)U(t)$$

where  $H(t)$  is the (possibly time dependent) self-adjoint energy operator (Hamiltonian) and  $\hbar$  the Planck constant.

### 4 Space

There exists the three-dimensional Euclidean space  $\mathbf{R}^3$ . Its group of motion is formed by the translation group  $T^3$  and the rotation group  $O(3)$ . One supposes that in the Hilbert space  $\mathcal{H}$  there is a unitary representation  $U(a)$  of the translation by the three-vector  $a$ . If  $(\Omega, \mathcal{F})$  is a measured space and  $\{E_B, B \in \mathcal{F}\}$  is the associated POVM then one has

$$U(a)E_B U(a)^* = E_{\alpha_a(B)}$$

where  $\alpha_a : \mathcal{F} \rightarrow \mathcal{F}$  is the group of automorphisms.

One has also a projective representation of the rotation group  $SO(3)$  which can be made into a unitary representation  $U(R)$  of the covering group  $SU(2)$ , here  $R \in SU(2)$ . Hopefully the distinction by the type of argument of  $U$  will be sufficient to avoid confusion. The irreducible representations of  $SU(2)$  describes systems with integer and half-integer spins.

### 5 Composite systems

If there are two different systems with assigned Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  then the composite system is described by the tensor product

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

## 6 Bose-Fermi alternative

The Hilbert space of an  $N$ -particle system is the  $N$ -fold tensor product of the single particle Hilbert spaces provided that the particles are not of the same species. For identical particles with integer spin (bosons) one uses the symmetrized  $N$ -fold tensor product  $(\mathcal{H}^{\otimes N})_S$  of the Hilbert space  $\mathcal{H}$ . For identical particles with half integer spin (fermions) one uses the anti-symmetrized  $N$ -fold tensor product  $(\mathcal{H}^{\otimes N})_A$ .

## 7 Internal symmetries

There is a compact Lie group  $G_{int}$  of internal symmetries and its unitary representation  $U(\tau), \tau \in G_{int}$  in the Hilbert space  $\mathcal{H}$  which commutes with representations of the translation group  $U(a)$  and the rotation group  $U(R)$ . For instance one could have the gauge group  $G_{int} = U(1)$  which describes the electric charge. The group generates the superselection sectors.

### SUMMARY

Axiomatic quantum mechanics described by the presented seven axioms can be briefly formulated as follows. There is space and time  $\mathbf{R}^1 \times \mathbf{R}^3$ , the symmetry group  $G = T^1 \times T^3 \times SU(2) \times G_{int}$ , the Hilbert space  $\mathcal{H}$  and the unitary representation  $U(g)$  of  $G$ , here  $g \in G$ . Axiomatic quantum mechanics is given by the following data:

$$\{\mathcal{H}, U(g), \rho, (\Omega, \mathcal{F}, \alpha_g), \{E_B, \Gamma_B, B \in \mathcal{F}\}\}$$

Here  $\rho$  is the density operator,  $(\Omega, \mathcal{F})$  is the measured space,  $\alpha_g$  is the group of automorphisms of the  $\sigma$ -algebra  $\mathcal{F}$ ,  $\{E_B\}$  is POVM, and  $\{\Gamma_B\}$  the state transformer.

**Example.** An example of quantum system satisfying the all seven axioms is given by the non-relativistic spin one half particle with the Hilbert space  $\mathcal{H} = \mathbf{C}^2 \otimes L^2(\mathbf{R}^3)$  and the Schrödinger-Pauli Hamiltonian and also by its multi-particle generalization.

### COMMENTS

We can add more axioms, of course. In particular we did not postulate yet the covariance under the Poincare or Galilei group (for the Galilei group one has a projective representation) but only invariance under spatial translations and rotations which we have in the non-relativistic theory as well as in the relativistic theory. We could add also the condition of the positivity of energy. Finally, we could postulate the standard non-relativistic Schrödinger equation for  $N$  bodies as a fundamental axiom of quantum mechanics. Note also that in relativistic quantum field theory all the axioms are valid (in fact we have to add more axioms to get quantum field theory) except the second axiom on measurements which requires a special discussion.

Note in the conclusion that the spatial approach to quantum mechanics explicitly formulated in this note was used for the investigation of quantum non-locality. It was shown in [16],[20] that quantum non-locality in the sense of Bell there exists only because the spatial

properties of quantum system were neglected. If we take the spatial degrees of freedom into account then local realism might be consistent with quantum mechanics and with performed experiments. If somebody wants to depart from local realism then he/she has to change quantum mechanics. The local realist representation in quantum mechanics was formulated in [20] as follows:

$$Tr(\rho A_m(x) B_n(y)) = E\xi_m(x)\eta_n(y)$$

Here  $A$  and  $B$  are observables depending on the space points  $x$  and  $y$  and on parameters  $m$  and  $n$  while  $E$  is the expectation of two random fields  $\xi_m(x)$  and  $\eta_n(y)$ . The representation was proved in [20] under some rather restrictive assumptions. It would be important to prove the representation under more general assumptions. The non-commutative spectral theory related with the local realist representation is discussed in [20], definitions of local realism in the sense of Bell and in the sense of Einstein and its relations with the contextual approach are considered in [21].

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